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Infinitesimal change of stable basis

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Abstract. The purpose of this note is to study the Maulik-Okounkov $K-$theoretic stable basis for the Hilbert scheme of points on the plane, which depends on a "slope" $m \in \mathbb{R}$. When $m = \frac{a}{b}$ is rational, we study the change of stable basis from slope $m - \varepsilon$ to $m + \varepsilon$ for small $\varepsilon > 0$, and conjecture that it is related to the Leclerc-Thibon conjugation in the $q-$Fock space for $U_q\widehat{\mathfrak{gl}}_b$. This is part of a wide framework of connections involving derived categories of quantized Hilbert schemes, modules for rational Cherednik algebras and Hecke algebras at roots of unity.

Résumé. Le but de cette note est d'étudier la base stable $K-$théoretique de Maulik-Okounkov pour la schéma de Hilbert de points sur le plan, qui depend d'une pente $m \in \mathbb{R}$. Quand $m = \frac{a}{b}$ est rationelle, nous étudions le changement de base stable de la pente $m - \varepsilon$ à $m + \varepsilon$ pour un petit $\varepsilon > 0$, et conjecturons qu'il est lié à la conjugaison de Leclerc-Thibon sur le $q-$éspace de Fock pour $U_q\widehat{\mathfrak{gl}}_b$. C'est une partie d'un cadre large des connections entre les catégories dérivées des schémas de Hilbert quantiques, les modules sur les algèbres de Cherednik rationelles, et les algèbres de Hecke aux racines de l'unité.

Keywords: Stable and canonical bases, Leclerc-Thibon involution, Hilbert schemes

1 Introduction

Maulik and Okounkov [8, 7] developed a new paradigm for constructing interesting bases (called **stable bases**) in the equivariant cohomology and K-theory of certain algebraic varieties with torus actions. In this extended abstract, we present an explicit conjectural description of the K–theoretic stable bases for Hilb_n, the Hilbert scheme of n points on $\mathbb{C}^2$.

The definition of the stable basis involves a choice of a Hamiltonian one parameter subgroup, which is unique in the case of Hilbert schemes (strictly speaking, there are two possible choices since one can invert the parameter, but we fix it without loss of generality), and a choice of $\mathcal{L} \in \text{Pic}(\text{Hilb}_n) \otimes (\mathbb{R} \backslash \mathbb{Q})$. We abuse notation and refer

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to such $\mathcal{L}$ as "line bundles", though they are formal irrational multiples of actual line bundles. Since $\text{Pic}(\text{Hilb}_n)$ has rank 1 with generator $\mathcal{O}(1)$, we write $\mathcal{L}_m$ for the line bundle associated to $m \in \mathbb{R}\backslash\mathbb{Q}$. The construction of [7] produces a basis:

$$\left\{s_\lambda^m\right\}_{\lambda \vdash n} \in K_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Hilb}_n) \qquad \forall\ m \in \mathbb{R}\backslash\mathbb{Q}. \tag{1.1}$$

For $m = 0$ the basis s^m is expected to match the (plethystically transformed) Schur polynomial basis, and for $m = \infty$ it coincides with the (modified) Macdonald polynomial basis. Therefore, the stable basis for general m can be thought of as interpolating between the bases of Schur and Macdonald polynomials. We are interested in "walls", i.e. those:

$$m \in \mathbb{R} \qquad \text{such that} \qquad \left\{s_\lambda^{m+\varepsilon}\right\}_{\lambda \vdash n} \neq \left\{s_\lambda^{m-\varepsilon}\right\}_{\lambda \vdash n}.$$

Throughout this paper, ε denotes a very small positive real number. There are only discretely many walls for each fixed n, all expected to be of the form $m = \frac{a}{b}$ with $0 < b \leq n$. The following conjecture prescribes how the stable basis changes upon crossing these walls:

Conjecture 1.2. *(see Conjecture 5.7 for the precise formulation): For $m = \frac{a}{b}$ with $\gcd(a, b) = 1$ the matrix taking $\left\{s_\lambda^{m+\varepsilon}\right\}_{\lambda \vdash n}$ to $\left\{s_\lambda^{m-\varepsilon}\right\}_{\lambda \vdash n}$ coincides with the Leclerc-Thibon involution [5, 6] for $U_q\widehat{\mathfrak{gl}}_b$, up to conjugation by the diagonal matrix that produces the renormalization (5.6).*

We prove the above conjecture for $b = 1$, where the Leclerc-Thibon involution is trivial:

Proposition 1.3. *We have $s_\lambda^\varepsilon = s_\lambda^{-\varepsilon}$ for all partitions $\lambda \vdash n$.*

The proof of Proposition 1.3, as well as an idea to tackle Conjecture 1.2 in general, is based on a principle that goes back to the work of Grojnowski and Nakajima, which says that one should work with all Hilb_n together, for all $n \in \mathbb{N}$. Namely, define: $K = \bigoplus_{n=0}^\infty K_{\mathbb{C}^* \times \mathbb{C}^*}(\text{Hilb}_n)$. Feigin-Tsymbaliuk [2] and Schiffmann-Vasserot [13] have constructed an action of the spherical double affine Hecke algebra (DAHA) $\mathcal{A}$ of type GL_∞ on K, albeit each in a different language. The algebra $\mathcal{A}$ has numerous q–Heisenberg subalgebras $\mathcal{A}^{(m)}$, parametrized by rational numbers m. In previous work ([12, 11]) the second named author proved that the action of $\mathcal{A}^{(m)}$, written in the stable basis s^m, is given by ribbon tableau formulas akin to those studied by Lascoux, Leclerc and Thibon [4]. We conjecture that this is a special case of the following more general phenomenon.

Conjecture 1.4. *(see Conjecture 6.3 for the precise formulation): For $m = \frac{a}{b}$ with $\gcd(a, b) = 1$ there exists an action $U_q\widehat{\mathfrak{gl}}_b \curvearrowright K$ such that:*

1. *K is a level 1 vacuum module for $U_q\widehat{\mathfrak{gl}}_b$, isomorphic to the Fock space.*

2. *The subalgebra $\mathcal{A}^{(m)}$ embeds into $U_q\widehat{\mathfrak{gl}}_b$ as the standard diagonal q–Heisenberg subalgebra, and this embedding intertwines its action on K from [2, 12, 13] with this action.*

3. *The bases $s^{m-\varepsilon}$ and $s^{m+\varepsilon}$ are, respectively, the standard and costandard bases for this action, up to renormalization.*

We expect that the above "slope m action" of $U_q\widehat{\mathfrak{gl}}_b$ on Fock space has interesting algebraic, geometric and combinatorial meaning, generalizing recent results about the "slope m action" $\mathcal{A}^{(m)} \curvearrowright K$ [1, 3, 10]. We support the conjectures with the following results.

Theorem 1.5. *Suppose that $\gcd(a,b) = \gcd(a',b) = 1$. Then the actions of $\mathcal{A}^{(\frac{a}{b})}$ and of $\mathcal{A}^{(\frac{a'}{b})}$ on K are conjugate to each other by the transition matrix between the bases $s^{\frac{a}{b}}$ and $s^{\frac{a'}{b}}$.*

Theorem 1.6. *Conjectures 1.2 and 1.4 are equivalent.*

Conjecture 1.2 was verified for $n \leq 6$ and all rational slopes $m = \frac{a}{b}$ by explicit computer calculations. Note that by (5.5), it is sufficient to check slopes $m \in [0,1)$ and by Proposition 5.8 one can assume $b \leq n(n-1)$. Therefore, one has finitely many slopes to check for each n.

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2 Symmetric functions

Much of the present paper is concerned with the ring of symmetric functions in infinitely many variables $x_1, x_2, \ldots$: $\Lambda = \mathbb{Z}[x_1, x_2, \ldots]^{\mathrm{Sym}}$. There are a number of generating sets of Λ, perhaps the most fundamental being the collection of monomial symmetric functions: $m_\lambda = \mathrm{Sym}\left(x_1^{\lambda_1} x_2^{\lambda_2} \ldots\right)$, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ goes over all partitions of natural numbers. Particular instances of monomial symmetric functions are the power sum functions: $p_k = m_{(k)} = x_1^k + x_2^k + \ldots$ and the elementary symmetric functions: $e_k = m_{(1,1,\ldots,1)} = \sum_{i_1 < \ldots < i_k} x_{i_1} \ldots x_{i_k}$. As a ring, Λ is generated by the elementary symmetric functions: $\Lambda = \mathbb{Z}[e_1, e_2, \ldots]$ and is generated by power sum functions upon tensoring with $\mathbb{Q}$: $\widetilde{\Lambda} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[p_1, p_2, \ldots]$. Additive generators are always indexed by partitions λ: $\Lambda = \mathbb{Z}[e_\lambda]_{\lambda \text{ partition}}$ where $e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots$ and: $\widetilde{\Lambda} = \mathbb{Q}[p_\lambda]_{\lambda \text{ partition}}$ where $p_\lambda =$

$p_{\lambda_1} p_{\lambda_2}$... A symmetric function is called **integral** if it lies in the image of $\Lambda \hookrightarrow \widetilde{\Lambda}$. A basis of $\widetilde{\Lambda}$ is called integral if it consists only of such functions.

There is a one-to-one correspondence between partitions and Young diagrams, the latter being stacks of 1×1 boxes placed in the corner of the first quadrant. For example, the Young diagram:

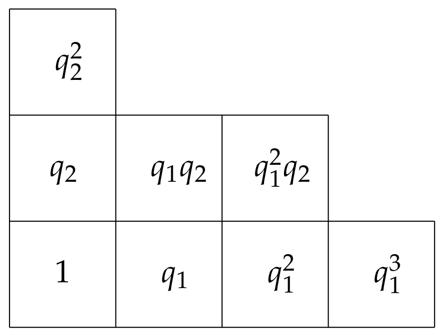

Figure 1

represents the partition $(4, 3, 1)$, because it has 4 boxes on the first row, 3 boxes on the second row, and 1 box on the third row. The monomials displayed in Figure 1 are called the **weights** of the boxes they are in, and are defined by the formula: $\chi_\square = q_1^x q_2^y$ where (x, y) are the coordinates of the southwest corner of the box in question. We call the integer: $c_\square = x - y$ the **content** of the box, and note that $c_\square$ is constant across diagonals. Finally, to every box in a Young diagram we may associate its **arm–length** and **leg–length**: $a(\square)$ and $l(\square) \in \mathbb{Z}_{\geq 0}$ These numbers count the distance between the box $\square$ and the right and top borders of the partition, respectively. For example, the box of weight q_2 in Figure 1 has $a(\square) = 2$ and $l(\square) = 1$. We will write:

$$c_\lambda = \sum_{\square \in \lambda} c_\square, \quad \chi_\lambda = \prod_{\square \in \lambda} \chi_\square. \tag{2.1}$$

We write $\mu \leq \lambda$ if the Young diagram of μ is completely contained in that of λ, and call $\lambda \backslash \mu$ a **skew Young diagram**. If such a skew diagram is a connected set of b boxes which contains no 2×2 squares, we call it a b–**ribbon**. Note that the contents of the boxes of a b–ribbon R are consecutive integers. Set:

$$\mathbf{h}(\text{ribbon } R) = \max_{\square, \blacksquare \in R} y(\square) - y(\blacksquare).$$

A skew diagram S is called a **horizontal** k–**strip** of b–ribbons if it can be tiled with k such ribbons $R_1, ..., R_k$ in such a way that the the northwestern most box of R_i does not lie below a box of R_j for any $1 \leq j \neq i \leq k$. Note that such a tiling of a skew diagram S is always unique. We set $\mathbf{h}(\text{strip } S) = \mathbf{h}(R_1) + ... + \mathbf{h}(R_k)$. The b–**core** of a partition λ is defined as the minimal partition which can be obtained by removing b–ribbons from λ. It is well known that the b–core does not depend on which set of ribbons we choose to remove, as long as this set is maximal.

We will now extend our rings of constants, and work instead with:

$$\Lambda_{q_1,q_2} = \Lambda \bigotimes_{\mathbb{Z}} \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}] = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}][x_1, x_2, ...]^{\text{Sym}}$$

$$\widetilde{\Lambda}_{q_1,q_2} = \widetilde{\Lambda} \bigotimes_{\mathbb{Q}} \mathbb{Q}(q_1, q_2) = \mathbb{Q}(q_1, q_2)[x_1, x_2, ...]^{\text{Sym}}.$$

The parameters q_1 and q_2 are normally denoted by q and t^{-1} in Macdonald polynomial theory. We choose to change the notation here, so as to not conflict with that of $q-$Fock spaces. Since the Macdonald inner product respects the degree of symmetric polynomials and the Hopf algebra structure of $\widetilde{\Lambda}_{q_1,q_2}$, it is uniquely determined by the pairing of p_k with itself:

$$\langle \cdot, \cdot \rangle_0 : \widetilde{\Lambda}_{q_1,q_2} \bigotimes_{\mathbb{Q}(q_1,q_2)} \widetilde{\Lambda}_{q_1,q_2} \longrightarrow \mathbb{Q}(q_1, q_2) \tag{2.2}$$

$$\langle p_k, p_k \rangle_0 = k \cdot \frac{1 - q_1^k}{1 - q_2^{-k}}$$

Macdonald polynomials $\{P_\lambda\}_{\lambda \text{ partition}}$ are the only orthogonal basis of $\widetilde{\Lambda}_{q_1,q_2}$:

$$\langle P_\lambda, P_\mu \rangle_0 = 0 \qquad \forall \ \lambda \neq \mu$$

which is unitriangular in the basis of monomial symmetric functions:

$$P_\lambda = m_\lambda + \sum_{\mu \lhd \lambda} m_\mu c_\lambda^\mu \tag{2.3}$$

for certain coefficients $c_\lambda^\mu \in \mathbb{Q}(q_1, q_2)$. In the above formula, recall that the **dominance ordering** on partitions of the same size $|\mu| = |\lambda|$ is:

$$\mu \unlhd \lambda \qquad \text{if} \qquad \mu_1 + ... + \mu_i \leq \lambda_1 + ... + \lambda_i \quad \forall i. \tag{2.4}$$

An element of $\widetilde{\Lambda}_{q_1,q_2}$ is called integral if it lies in the image of $\Lambda_{q_1,q_2} \hookrightarrow \widetilde{\Lambda}_{q_1,q_2}$. Because the coefficients c_λ^μ of (2.3) are rational functions in general, Macdonald polynomials are not integral. However, the following renormalization:

$$\widetilde{J}_\lambda = P_\lambda \cdot q_2^{-|\lambda|} \prod_{\square \in \lambda} \left(q_2^{l(\square)+1} - q_1^{a(\square)} \right) \tag{2.5}$$

is integral. It is well-known that the pairing of $\widetilde{J}_\lambda$ with itself is given by:

$$\langle \widetilde{J}_\lambda, \widetilde{J}_\mu \rangle_0 = \delta_\mu^\lambda \cdot q_2^{-|\lambda|} \prod_{\square \in \lambda} \left(q_2^{l(\square)+1} - q_1^{a(\square)} \right) \left(q_2^{l(\square)} - q_1^{a(\square)+1} \right). \tag{2.6}$$

3 Fock representation and global canonical bases

We recall the explicit construction of the action of the quantum affine algebra $U_q\widehat{\mathfrak{gl}}_b$ on the q–Fock space Λ_q, following Leclerc-Thibon [5, 6]. The **standard basis** in Λ_q will be denoted by $|\lambda\rangle$, so we define:

$$\Lambda_q = \bigoplus_{\lambda \text{ partition}} \mathbb{Q}(q) \cdot |\lambda\rangle.$$

Consider partitions λ, μ such that the former is obtained from the latter by adding an i**–node**, by which we mean a box $\blacksquare$ with content $\equiv i$ modulo b. We call this box a **removable** i**–node** for λ and an **indent** i**–node** for μ. Let $I_i(\mu)$ be the number of indent i–nodes of μ, $R_i(\lambda)$ the number of removable i–nodes of λ, $I_i^l(\lambda,\mu)$ (respectively $R_i^l(\lambda,\mu)$) the number of indent i–nodes (respectively of removable i–nodes) situated to the left of $\blacksquare$, and similarly, let $I_i^r(\lambda,\mu)$ and $R_i^r(\lambda,\mu)$be the corresponding numbers of nodes located on the right of $\blacksquare$. Set: $N_i(\lambda) = I_i(\lambda) - R_i(\lambda)$ for all partitions λ, as well as: $N_i^l(\lambda,\mu) = I_i^l(\lambda,\mu) - R_i^l(\lambda,\mu)$, $N_i^r(\lambda,\mu) = I_i^r(\lambda,\mu) - R_i^r(\lambda,\mu)$ for all pairs λ, μ such that $\lambda \backslash \mu$ consists of an i–node $\blacksquare$. Then the following assignments:

$$e_i|\lambda\rangle = \sum_{\text{an } i\text{–node}}^{\lambda/\mu \text{ is}} q^{N_i^l(\lambda,\mu)}|\mu\rangle, \qquad f_i|\mu\rangle = \sum_{\text{an } i\text{–node}}^{\lambda/\mu \text{ is}} q^{N_i^r(\lambda,\mu)}|\lambda\rangle, \tag{3.1}$$

$$q^{h_i}|\lambda\rangle = q^{N_i(\lambda)}|\lambda\rangle, \qquad q^D|\lambda\rangle = q^{N_0(\lambda)}|\lambda\rangle \tag{3.2}$$

give rise to an action of $U_q\widehat{\mathfrak{sl}}_b$ on the Fock space Λ_q. One wishes to enhance (3.1)–(3.2) to an action of: $U_q\widehat{\mathfrak{gl}}_b = U_q\widehat{\mathfrak{sl}}_b \otimes U_q\widehat{\mathfrak{gl}}_1$ on the Fock space, where the $q-$Heisenberg algebra is:

$$U_q\widehat{\mathfrak{gl}}_1 = \mathbb{Q}(q)\,\langle ..., B_{-2}, B_{-1}, B_1, B_2, ...\rangle \Big/ [B_k, B_l] - k\delta_{k+l}^0 [b]_{q^k}$$

where $[b]_x = 1 + x + ... + x^{b-1}$. In other words, we must define an action of the generators B_k on Fock space which commutes with the one prescribed by formulas (3.1)–(3.2). To do so, let us consider the following alternative system of generators:

$$\sum_{k=0}^{\infty} V_{\pm k} z^k = \exp\left(\sum_{k=1}^{\infty} \frac{B_{\mp k} z^k}{k}\right).$$

In [4], the authors introduced the following action $U_q\widehat{\mathfrak{gl}}_1 \curvearrowright \Lambda_q$ and showed that it commutes with the action of $U_q\widehat{\mathfrak{sl}}_b$ defined in (3.1)–(3.2), thus giving rise to an action $U_q\widehat{\mathfrak{gl}}_b \curvearrowright \Lambda_q$:

$$V_k|\mu\rangle = \sum_{\lambda} (-q)^{-\mathbf{h}(\lambda/\mu)}|\lambda\rangle, \qquad V_{-k}|\lambda\rangle = \sum_{\mu} (-q)^{-\mathbf{h}(\lambda/\mu)}|\mu\rangle \tag{3.3}$$

where the sums go over all horizontal k–strips of b–ribbons λ/μ.

As observed by Leclerc and Thibon, there is a unique involution of the Fock space Λ_q satisfying:

1. Semilinearity: $\overline{a(q)x + b(q)y} = a(q^{-1})\overline{x} + b(q^{-1})\overline{y}$.

2. Identity on vacuum: $\overline{|\varnothing\rangle} = |\varnothing\rangle$.

3. Invariance under the creation operators: $\overline{f_i v} = f_i \overline{v}$, $\overline{B_{-k} v} = B_{-k}\overline{v}$.

Indeed, products of f_i and B_{-k} applied to the vacuum span the Fock space, and this implies uniqueness. Note that $\overline{V_k v} = V_k \overline{v}$ for all $k > 0$, because the operators V_k are monomials in the generators B_{-k} with constant coefficients. Define the matrix $A_b(q) = (a_\lambda^\mu(q))$ by the equation

$$\overline{|\lambda\rangle} = \sum_\mu a_\lambda^\mu(q) \cdot |\mu\rangle. \tag{3.4}$$

Clearly, $A_b(q)A_b(q^{-1}) = \mathrm{Id}$ by the semilinearity property (1).

Theorem 3.5. ([5, 6]) *The matrix $A_b(q)$ has the following properties:*

a) $a_\lambda^\mu(q) \in \mathbb{Z}[q, q^{-1}]$.

b) $a_\lambda^\mu(q) = 0$ *unless* $|\lambda| = |\mu|$, $\mu \trianglelefteq \lambda$ *and* λ, μ *have the same b–core.*

c) $a_\lambda^\lambda(q) = 1$.

d) $a_\lambda^\mu(q) = a_{\mu'}^{\lambda'}(q)$.

We will also encounter the **costandard basis** $\overline{|\lambda\rangle}$ of Λ_q. By definition, $A_b(q)$ is the transition matrix between the standard and the costandard bases. Furthermore, the action of the creation operators in the costandard basis is given by the following equations:

$$f_i \overline{|\mu\rangle} = \overline{f_i |\mu\rangle} = \overline{\sum_\lambda q^{N_i^r(\lambda,\mu)} |\lambda\rangle} = \sum_\lambda q^{-N_i^r(\lambda,\mu)} \overline{|\lambda\rangle}, \tag{3.6}$$

and similarly:

$$V_k \overline{|\mu\rangle} = \sum_\lambda (-q)^{\mathbf{h}(\lambda\backslash\mu)} \overline{|\lambda\rangle}, \tag{3.7}$$

where the sums over λ and μ are the same as in (3.1) and (3.3).

4 Hilbert schemes

We consider the Hilbert scheme Hilb_n of n points in the plane. This is a smooth quasi-projective variety of dimension $2n$. It is endowed with a torus action: $T = \mathbb{C}^*_q \times \mathbb{C}^*_t \curvearrowright \text{Hilb}_n$. In the above formula, q and t are equivariant parameters, namely the standard coordinates on rank 1 tori. We will often denote $q_1 = qt$ and $q_2 = qt^{-1}$ and think of these monomials as the torus characters acting on the coordinate lines of $\mathbb{C}^2$. Fixed points of the Hilbert scheme with respect to the torus action are monomial ideals:

$$I_\lambda = (x^{\lambda_1 - 1}, x^{\lambda_2 - 1}y, x^{\lambda_3 - 2}y^2, ...) \in \text{Hilb}_n \tag{4.1}$$

for any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ...)$. The torus character in the tangent space to Hilb_n at the fixed point I_λ is given by the well-known formula:

$$T_\lambda \text{Hilb}_n = \sum_{\square \in \lambda} \left(q_1^{a(\square)} q_2^{-l(\square)-1} + q_1^{-a(\square)-1} q_2^{l(\square)} \right). \tag{4.2}$$

We will work with the equivariant $K-$theory group: $K = \bigoplus_{n=0}^{\infty} K_{q,t}(\text{Hilb}_n)$. Important elements of K are the skyscraper sheaves at the torus fixed points (4.1), which we denote by the same letter as the fixed point itself: $[\widetilde{I}_\lambda] \in K$. Recall the equivariant localization formula, which expresses any class $f \in K$ in terms of its restrictions to torus fixed points:

$$f = \sum_{\lambda \vdash n} \frac{f|_\lambda \cdot [\widetilde{I}_\lambda]}{[T_\lambda \text{Hilb}_n]} \tag{4.3}$$

where in the denominator we write $[x] = 1 - x^{-1}$ and extend this notation additively: $[x + y] = [x] \cdot [y]$. Because of the presence of denominators, the equality (4.3) holds in the localized K–theory group: $\widetilde{K} = K \bigotimes_{\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]} \mathbb{Q}(q_1, q_2)$ In this localization, we may renormalize the classes of fixed points:

$$[I_\lambda] = \frac{[\widetilde{I}_\lambda]}{[T_\lambda \text{Hilb}_n]} \in \widetilde{K}.$$

The restriction of a class to a fixed point is precisely its coefficient when expanded in the basis $[I_\lambda]$: $f = \sum_{\lambda \vdash n} f|_\lambda \cdot [I_\lambda]$.

Haiman showed one can identify $K \cong \Lambda_{q_1,q_2}$ such that the classes of fixed points correspond to **modified Macdonald polynomials** $\widetilde{H}_\lambda$: $[\widetilde{I}_\lambda] \leftrightarrow \widetilde{H}_\lambda$ where $\widetilde{H}_\lambda[X] = \widetilde{J}_\lambda \left[\frac{X}{1 - q_2^{-1}} \right]$ is the image of (2.5) under the standard plethysm. The Bergeron–Garsia operator ∇ is defined to be diagonal in the basis of modified Macdonald polynomials:

$$\nabla : \Lambda_{q_1,q_2} \longrightarrow \Lambda_{q_1,q_2}, \qquad \widetilde{H}_\lambda \mapsto \widetilde{H}_\lambda \cdot \chi_\lambda$$

where χ_λ was defined in (2.1). If we observe that χ_λ is the torus weight of the restriction of the line bundle $\mathcal{O}(1)$ to the fixed point λ, then the operator ∇ corresponds to the operator of multiplication by $\mathcal{O}(1)$.

5 Stable bases

In [8], Maulik and Okounkov defined the **stable basis** for the cohomology of a wide class of symplectic resolutions X. The $K-$theoretic version of their construction has not yet been published, but the interested reader can read about it in [10, 11]. We will review their particular construction in the case at hand $X = \text{Hilb}_n$:

$$\forall\ m \in \mathbb{R}\backslash\mathbb{Q} \quad \rightsquigarrow \quad \text{an integral basis } \{s^m_\lambda\}_{\lambda\vdash n} \in K_T(\text{Hilb}_n) \tag{5.1}$$

which is triangular in terms of renormalized fixed points:

$$s^m_\lambda = \sum_{\mu \trianglelefteq \lambda} \gamma^\mu_\lambda [I_\mu] \qquad \text{where} \qquad \gamma^\lambda_\lambda = \prod_{\square\in\lambda} \left(q_2^{l(\square)} - q_1^{a(\square)+1}\right) \tag{5.2}$$

and the coefficients $\gamma^\mu_\lambda \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ have the property:

$$\text{min deg } \gamma^\mu_\lambda(q,t) \geq -n(\mu) + m\cdot(c_\mu - c_\lambda) \tag{5.3}$$

$$\text{max deg } \gamma^\mu_\lambda(q,t) \leq n(\mu') + |\mu| + m\cdot(c_\mu - c_\lambda). \tag{5.4}$$

Recall that $n(\lambda) = \sum_{\square\in\lambda} l(\square)$. Here and throughout this paper, “min deg" and “max deg" refer to the minimal and maximal degrees of a Laurent polynomial in the variable t. Formulas (5.3)–(5.4) are arranged so that when $\lambda = \mu$, the leading coefficient of (5.2) forces the two inequalities to be equalities. Maulik–Okounkov claim that for any $m \in \mathbb{R}\backslash\mathbb{Q}$, there is a unique integral basis with properties (5.2), (5.3), (5.4). Moreover, the basis is unchanged under small perturbations of m. Note that uniqueness implies:

$$s^{m+1}_\lambda = \frac{\nabla s^m_\lambda}{\chi_\lambda}. \tag{5.5}$$

The existence and uniqueness of (5.1) also holds for $m \in \mathbb{Q}$, but we must require either (5.3) or (5.4) to be a strict inequality. Fix a rational slope $m \in \mathbb{Q}$. Since the stable basis is locally constant on a small punctured neighborhood of m, we have the two different bases:

$$\{s^{m-\varepsilon}_\lambda\}_{\lambda \text{ partition}} \subset \Lambda_{q_1,q_2} \supset \{s^{m+\varepsilon}_\lambda\}_{\lambda \text{ partition}}.$$

Our main object of study will be the transition matrix between the above stable bases:

$$A : \Lambda_{q_1,q_2} \longrightarrow \Lambda_{q_1,q_2}, \quad A\left(s^{m+\varepsilon}_\lambda\right) = s^{m-\varepsilon}_\lambda$$

for all partitions λ. When $m = \frac{a}{b}$ with $\gcd(a,b) = 1$, we will relate the matrix A with the representation theory of $U_q\widehat{\mathfrak{gl}}_b$, as in Section 3. Specifically, we consider the renormalized stable basis given by:

$$\tilde{s}^{m\pm\varepsilon}_\lambda = s^{m\pm\varepsilon}_\lambda \cdot o^m_\lambda \cdot \prod_{\substack{\lambda\backslash\text{core }\lambda\, = \\ =\, R_1\sqcup\ldots\sqcup R_k}} \prod_{i=1}^{k}\prod_{j=1}^{b-1} q^{\#^i_j} \tag{5.6}$$

where the product is taken over any maximal set of $b-$ribbons contained in λ, and:

$$\#_j^i = \begin{cases} mj - \lfloor mj \rfloor & \text{if the } j-\text{th step in the ribbon } R_i \text{ is to the right} \\ \lceil mj \rceil - mj & \text{if the } j-\text{th step in the ribbon } R_i \text{ is down.} \end{cases}$$

Conjecture 5.7. *In the renormalized stable basis, we have:*

$$\tilde{s}_\lambda^{m-\varepsilon} = A\left(\tilde{s}_\lambda^{m+\varepsilon}\right) = \sum_\mu a_\lambda^\mu(q) \cdot \tilde{s}_\mu^{m+\varepsilon}$$

where $(a_\lambda^\mu(q))$ *is the matrix of the Leclerc-Thibon involution* (3.4).

It is clear from the definition that the stable bases are locally constant in the parameter m. More precisely, we say that the stable basis for Hilb_n has a **wall** at m if $s^{m-\varepsilon} \neq s^{m+\varepsilon}$ for some small $\varepsilon > 0$.

Proposition 5.8. *If* $m = \frac{a}{b}$ *with* $\gcd(a, b) = 1$ *is a wall for* Hilb_n*, then the following statements hold:*

a) $b \leq n(n-1)$.

b) The transition matrix between $s^{m+\varepsilon}$ *and* $s^{m-\varepsilon}$ *is block-triangular. Two partitions* λ *and* μ *belong to the same block if* $m \cdot (c_\lambda - c_\mu) \in \mathbb{Z}$.

Proof. Since $|c_\lambda|, |c_\mu| \leq \frac{n(n-1)}{2}$, we conclude that:

$$b \leq c_\lambda - c_\mu \leq n(n-1).$$

which implies (a). Part (b) is immediate from equations (5.3) and (5.4). □

Conjecture 5.7 implies stronger constraints on the set of walls than Proposition 5.8 does, and it also refines the blocks in the the wall-crossing matrices:

Proposition 5.9. *Assume that Conjecture 5.7 holds and* $m = \frac{a}{b}$ *is a wall for Hilb*$_n$, $\gcd(a, b) = 1$*. Then the following statements hold:*

a) $b \leq n$.

b) The transition matrix between $s^{m+\varepsilon}$ *and* $s^{m-\varepsilon}$ *is block-triangular. Two partitions* λ *and* μ *belong to the same block if they have the same b–core.*

Proof. Part (b) follows from Theorem 3.5 (b). Suppose for the purpose of contradiction that $b > n$. Then every partition of n is its own b–core, so all blocks are of size 1. Since the transition matrix should have 1's on the diagonal, it is an identity matrix, and therefore $m = \frac{a}{b}$ is not a wall. □

6 Heisenberg actions

To prove Conjecture 5.7, for each $m = \frac{a}{b}$ one needs to present an action of $U_q\widehat{\mathfrak{gl}}_b$ on the Fock space such that the matrices of the generators in the renormalized stable bases $\tilde{s}^{m-\varepsilon}$ and $\tilde{s}^{m+\varepsilon}$ have particularly nice form. In this section, we present such an action of the diagonal Heisenberg subalgebra: $U_q\widehat{\mathfrak{gl}}_1 \subset U_q\widehat{\mathfrak{gl}}_b$ following [12]. We will use a remarkable algebra $\mathcal{A}$ over the field $\mathbb{Q}(q,t)$, which is known by many names: the double shuffle algebra, the Hall algebra of an elliptic curve, the doubly-deformed $W_{1+\infty}$–algebra, the spherical double affine Hecke algebra (DAHA) of type GL_∞. See [13, 9] for various isomorphisms between different presentations of $\mathcal{A}$. It is known that the group $SL(2,\mathbb{Z})$ acts on $\mathcal{A}$ by automorphisms. Furthermore, there is a natural q–Heisenberg subalgebra of $\mathcal{A}$, which in the DAHA presentation is generated by symmetric polynomials in X_i and their conjugates. By applying automorphisms $\gamma \in SL(2,\mathbb{Z})$ to this subalgebra, we get new q–Heisenberg subalgebras:

$$\mathcal{A} \supset \mathcal{A}^{(m)} = \mathbb{Q}(q,t)\left\langle ..., B^{(m)}_{-2}, B^{(m)}_{-1}, B^{(m)}_{1}, B^{(m)}_{2}, ...\right\rangle$$

labeled by rational numbers $m = a/b$, where $\gamma(1,0) = (b,a)$. We will call $\mathcal{A}^{(m)}$ the **slope** m **subalgebra** in $\mathcal{A}$. The following results relate $\mathcal{A}^{(m)}$ to slope m stable bases.

Theorem 6.1. ([2, 13, 10]) *There is an action of $\mathcal{A}$ on Λ_{q_1,q_2}, where $q_1 = qt$ and $q_2 = qt^{-1}$.*

Theorem 6.2. ([12]) *The action of the slope m subalgebra $\mathcal{A}^{(m)}$ in the renormalized stable basis $\tilde{s}^{m+\varepsilon}$ is given by equations* (3.3).

Conjecture 5.7 can be now reformulated in the following way, which is more interesting for geometric applications.

Conjecture 6.3. *Given $m = \frac{a}{b}$ with $\gcd(a,b) = 1$, there is an action of the quantum affine algebra $U_q\widehat{\mathfrak{sl}}_b$ on the Fock space, satisfying the following conditions:*

a) It commutes with the action of the slope m Heisenberg subalgebra $\mathcal{A}^{(m)}$.

b) The action of the creation operators f_i in the renormalized stable basis $\tilde{s}^{m+\varepsilon}$ is given by (3.6).

c) The action of the creation operators f_i in the renormalized stable basis $\tilde{s}^{m-\varepsilon}$ is given by (3.1).

Theorem 6.4. *Conjectures 5.7 and 6.3 are equivalent.*

Based on the extensive computer experiments, we formulate the following conjecture.

Conjecture 6.5. *For all positive slopes m, the stable basis is Schur-positive:*

$$s^m_\lambda = \sum_\mu k^m_{\lambda,\mu} s_\mu, \qquad k^m_{\lambda,\mu} \in \mathbb{N}[[q,t]].$$

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